Article received on October 16th 2017
Article accepted on November 13th 2017
UDC: 78.02
51-78:781.1/.4

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MUSICA FIBONACCIANA: AESTHETIC AND PRACTICAL APPROACH

Abstract: In the sphere of musical research, the intersection of two seemingly very different subject areas – music and mathematics – is in essence related to one of the trends of music – attributing the theory of music to science, to the sphere of *mathematica*. It is regarded the longest lasting interdisciplinary dialogue. The implication of numerical proportions and number sequences in music composition of different epochs is closely related to this sphere. A significant role in creating music was attributed to the so-called infinite Fibonacci sequence. Perhaps the most important feature of the Fibonacci numbers, which attracted the attention of thinkers and creators of different epochs, is the fact that by means of the ratio between them it is possible to come maximally close to the Golden Ratio formula, which expresses the laws of nature. On a practical plane, often the climax, the most important part of any composition, matches the point of the Golden Ratio; groups of notes, rhythm, choice of tone pitches, grouping of measures, time signature, as well as proportions between a musical composition's parts may be regulated according to Fibonacci principles. The article presents three analytical cases – Chopin's piano prelude, Bourgeois' composition for organ and Reich's minimalistic piece, attempting to render music composition structure to the logic of Fibonacci numbers.
Key words: numerical proportions, Fibonacci sequence in music, Frederic Chopin, Derek Bourgeois, Steve Reich

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Introduction: Numerical Proportions in the Concept of Harmony of the Spheres

The idea of the art of sound being based on mathematics may be referred to by Gioseffo Zarlino as *numero sonoro*.[1] St. Thomas Aquinas' phrase deserves mention within this context too – "music, which studies the ratios of audible sounds."[2] The roots of this idea go back to Antiquity and the mathematical substantiation of the universe, as, for example, Aristotle stated: the criteria of beauty, "orderly arrangement, proportion, and definiteness", are "especially manifested by the mathematical sciences."[3]

From the perspective of a European perception, the Pythagoreans were already regarded to be the first to raise the issue of uniting music and mathematics, sound and number. They perceived music as an abstract sphere based on mathematical means, incorporating music into the universal harmony of numerical proportions.[4] Over the course of later epochs, this kind of music theory has been classified as Latin *musica theorica* | *theoretica* | *contemplativa* | *speculativa* | *arithmetica*, etc. In the first dictionary of music written in the German language, *Musikalisches Lexikon oder musikalische Bibliothek* (1732), among different types of music, the type *musica arithmetica* is indicated, that is, arranging the sounds in proportions and numbers.[5]

Considering the importance of proportions in music compositions of different epochs we should get back to Antiquity, where the environment (cosmos) that surrounded man was treated as a creation of God. The cosmos was an example of perfect order or harmony due to the characteristics of regularities, symmetry and proportions, which were expressed in certain numerical relations.[6] This ex-

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1 For the discussion on music as a mathematical science Zarlino devoted a chapter "Per qual cagione la Musica sia detta subalternata all'Arithmetica, & mezanatra la mathematica, & la naturale" (Cap. 20) in his famous treatise *The Harmonic Institutions* (1558).

2 "Musica, quae considerat proportiones sonorum audibilium" was written in St. Thomas Aquinas' *Commentary on Aristotle's Metaphysics*, Bk. 3 Lsn 7 Sct 412, transl. John P. Rowan, Chicago, 1961, 201.

3 The definition was presented by Aristotle in his *Metaphysics* (4th c. BC), Book 13, Part 3, 1078a–1078b (transl. Hugh Tredennick); also quoted in Albert L. Blackwell's *The Sacred in Music*, Westminster John Knox Press, 2000, 162.

4 It is stated that Pythagoras said: "There is geometry in the humming of strings. There is music in the spacing of the spheres." The quotation was cited in: Louise B. Young's *The Mystery of Matter*, Oxford University Press, 1965, 113.

5 "Musica Arithmetica [lat.ital.] Musique Arithmetique [gall.] betrachtet die Klänge nach der Proportion, so sie mit den Zahlen Machen." From: Johann Gottfried Walther, *Musikalisches Lexikon oder musikalische Bibliothek*, Leipzig, W. Deer, 1732, 431.

6 This is encoded in the very meaning of the word "cosmos": Greek *cosmos*/*κόσμος* – order, orderly arrangement, a harmonious system.

planation of universal harmony in terms of numbers, and making mathematical principles absolute, determined the originality of the aesthetics of the Pythagoreans. For example, this is testified to by the work written in c. 361 BC – Plato's dialogue *Timaeus* – a work about a mathematically calculated idea of beauty and the importance of numerical relations to the creative process. This literary source talks about the harmony of the cosmos that is based on the relations between three proportions (arithmetic, geometric and harmonic),[7] because Demiurge, the Divine Craftsman, "made" the cosmic soul from a mixture divided according to the algorithms of the cosmic *septenarius* 1–2–3–4–8–9–27 and three proportions:

> And he proceeded to divide in this manner: – First of all, he took away one part of the whole [1], and then he separated a second part which was double the first [2], and then he took away a third part which was half as much again as the second and three times as much as the first [3], and then he took a fourth part which was twice as much as the second [4], and a fifth part which was three times the third [9], and a sixth part which was eight times the first [8], and a seventh part which was twenty-seven times the first [27]. After this he filled up the double intervals [i.e. between 1, 2, 4, 8] and the triple [i.e. between 1, 3, 9, 27] cutting off yet other portions from the mixture and placing them in the intervals, so that in each interval there were two kinds of means, the one exceeding and exceeded by equal parts of its extremes [as for example 1, 4/3, 2, in which the mean 4/3 is one-third of 1 more than 1, and one-third of 2 less than 2], the other being that kind of mean which exceeds and is exceeded by an equal number. Where there were intervals of 3/2 and of 4/3 and of 9/8, made by the connecting terms in the former intervals, he filled up all the intervals of 4/3 with the interval of 9/8, leaving a fraction over; and the interval which this fraction expressed was in the ratio of 256 to 243.[8]

[7] The three proportions became one of the major principles of beauty (or harmony) and attracted attention for their logical nature – a consistent "growing". In arithmetic proportion, this growing is represented accordingly: the second dimension is larger than the first one as much as the third dimension is larger than the second, 1 : 1 ½ : 2 or 1 : 2 : 3. This is the simplest proportion of numbers:

0–1–2–3–4–5–6–7–8–9… or 0–2–4–6–8–10–12–14 …

In geometric proportion, the ratio between the second and the first magnitudes coincides with that of the third and the second magnitudes: 1 : 2 : 4, because 1 : 2 = 2 : 4. The progression of numbers is formed accordingly:

1–2–4–8–16–32–64–128 …

A complicated harmonic proportion expresses the relationship between the following three numbers: a, $2ab : (a+b)$, and b, where the third number is larger than the second one by such a part of its size as the second number is larger than the first one by the same part of the size of the first. For example:

1 : 4/3 : 2 or 3 : 4 : 6.

[8] Quote from Plato's *Timaeus*, 35b–36b, transl. by Benjamin Jowett, published in: *The Dialogues of Plato*, Vol. 3: *The Republic, Timaeus, Critias,* 3rd ed., Oxford University Press, London, Humphrey Milford, 1892, 36).

Having compared the creative process described by Plato and the Pythagorean system of numerical relations of musical intervals, it becomes clear that music participated inseparably in the creation of the world. This is because the proportions of "decomposing the mixture" used in Demiurge's work are identical to the numerical relations of musical intervals described by the Pythagoreans:

2 : 1 as the perfect octave,
3 : 2 as the perfect fifth,
4 : 3 as the perfect fourth,
9 : 8 (3/2 : 4/3) as the whole tone.[9]

[9] The history of a numerical substantiation of musical intervals is rephrased by a legend about Pythagoras. Here it is a retelling by Manly Palmer Hall in 1928 in his book *The Secret Teachings of All Ages*:

> One day while meditating upon the problem of harmony, Pythagoras chanced to pass a brazier's shop where workmen were pounding out a piece of metal upon an anvil. By noting the variances in pitch between the sounds made by large hammers and those made by smaller implements, and carefully estimating the harmonies and discords resulting from combinations of these sounds, he gained his first clue to the musical intervals of the diatonic scale. He entered the shop, and after carefully examining the tools and making mental note of their weights, returned to his own house and constructed an arm of wood so that it: extended out from the wall of his room. At regular intervals along this arm he attached four cords, all of like composition, size, and weight. To the first of these he attached a twelve-pound weight, to the second a nine-pound weight, to the third an eight-pound weight, and to the fourth a six-pound weight. These different weights corresponded to the sizes of the braziers' hammers.
>
> Pythagoras thereupon discovered that the first and fourth strings when sounded together produced the harmonic interval of the octave, for doubling the weight had the same effect as halving the string. The tension of the first string being twice that of the fourth string, their ratio was said to be 2:1, or duple. By similar experimentation he ascertained that the first and third string produced the harmony of the diapente, or the interval of the fifth. The tension of the first string being half again as much as that of the third string, their ratio was said to be 3:2, or sesquialter. Likewise the second and fourth strings, having the same ratio as the first and third strings, yielded a diapente harmony. Continuing his investigation, Pythagoras discovered that the first and second strings produced the harmony of the diatessaron, or the interval of the third; and the tension of the first string being a third greater than that of the second string, their ratio was said to be 4:3, or sesquitercian. The third and fourth strings, having the same ratio as the first and second strings, produced another harmony of the diatessaron. According to Iamblichus, the second and third strings had the ratio of 8:9, or epogdoan.
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> From: Manly Palmer Hall, *The Secret Teachings of All Ages. An Encyclopedic Outline of Masonic, Hermetic, Qabbalistic and Rosicrucian Symbolical Philosophy*, Chapter 16 "The Pythagorean Theory of Music and Color", Los Angeles, Philosophical Research Society, 1928, 81, http://www.sacred-texts.com/eso/sta/sta19.htm [2017-09-07])

According to the Pythagoreans, these matches influenced the material proportions of the four elements – the primary conditions that were necessary for man's existence – earth, air, water and fire: earth was made up of four of its own parts, water was made up of three parts earth and one part fire, air was made up of three parts fire and one part earth, while fire was made up of four of its own parts. Moreover, in Antiquity the positioning of the known heavenly bodies (planets and stars) generated a harmonious sound, because their distances were related to the numerical relationships of musical intervals. This concept was named the Harmony of the Spheres.[10] With the reference to Macrobius and others, Manly P. Hall states, that according this concept the distance between the heavenly bodies created the "sound" of tones and semitones and the seven vowel letters of the Greek alphabet; in addition, each planet was compared to a concrete musical tone, number, color[11] and geometric form; each "sounded" one of the seven Greek modes (see Example 1).[12]

> The Pythagoreans believed that everything which existed had a voice and that all creatures were eternally singing the praise of the Creator. Man fails to hear these divine melodies because his soul is enmeshed in the illusion of material existence. When he liberates himself from the bondage of the lower world with its sense limitations, *the music of the spheres* will again be audible as it was in the Golden Age.[13]

The testimonies by ancient scholars (e.g. Iamblichus, Pliny, Irenaeus, Macrobius, Martianus Capella, etc.) provide various approaches to the same focus (for example, the equivalence of a particular planet to the music mode or concrete note). This confirms a rich diversity in forming and interpreting the ancient con-

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Heinrich Husmann described the creation process of musical intervals and their numerical equivalents using a one-string instrument called monochord. For example, if we put a finger on the half of string (thus, dividing the string in half or 2 : 1), we may hear the interval of the octave; if we put a finger on two thirds of the string (3 : 2) – the interval of the fifth, and so forth. For more information, see: Heinrich Husmann, *Grundlagen der Antiken und Orientalischen Musikkultur*, Berlin, Walter de Gruyter, 1961, 9–19.

[10] Other expressions meaning the universal harmony | universal music: German *Sphärenharmonie*, *Sphärenmusik*, English *Music of the Spheres*, Latin *musica universalis* (the concept *universus* [global, all in one] is adapted as an expression of the world, or the Cosmos).

[11] "The ancient city of Ecbatana as described by Herodotus, its seven walls colored according to the seven planets, revealed the knowledge of this subject possessed by the Persian Magi. The famous *zikkurat* or astronomical tower of the god Nebo at Borsippa ascended in seven great steps or stages, each step being painted in the key color of one of the planetary bodies." Manly Palmer Hall, op. cit., 84.

[12] Ibid., 82–83.

[13] Ibid., 83.

cept of the Harmony of the Spheres. Moreover, the vitality of this universal idea was so potent that it spread over the later epochs.[14]

Relations between 4 elements		
pair of elements	**formula**	**interval**
fire : earth	1 : 2	*perfect octave*
fire : air	1 : 4/3	*perfect fourth*
fire : water	1 : 3/2	*perfect fifth*
air : water	4/3 : 3/2	*whole tone*

Relations between heavenly bodies	
pair of planets	**interval**
Earth : Moon	*whole tone*
Moon : Mercury	*semitone*
Mercury : Venus	*semitone*
Venus : Sun	*whole & semitone*
Sun : Mars	*whole tone*
Mars : Jupiter	*semitone*
Jupiter : Saturn	*semitone*
Saturn : any star	*semitone*

sphere	**Greek vowel**
1st heaven	Α, α (*Alpha*)
2nd heaven	Ε, ε (*Epsilon*)
3rd heaven	Η, η (*Eta*)
4th heaven	Ι, ι (*Iota*)
5th heaven	Ο, ο (*Omicron*)
6th heaven	Υ, υ (*Upsilon*)
7th heaven	Ω, ω (*Omega*)

planet	**note**	**color**
Mars	do	red
Sun	re	orange
Mercury	mi	yellow
Saturn	fa	green
Jupiter	sol	blue
Venus	la	indigo
Moon	si (ti)	violet

Example 1: Illustration of relationships between the four elements, heavenly bodies, musical intervals, notes and colors[15]

[14] For example, Thomas Morley in his *Plaine and Easie Introduction to Practicall Musicke* (1597), created a chart that illustrated the perfect system of relationships between planets, goddesses, musical modes and Greek ideals of perfection. Robert Fludd presented a diagram of musical elements, repeating the Pythagorean musical intervals and the idea of harmonious relationship between the four elements (*De Musica Mundana*, 1618). In *Harmonices mundi* (1619) Johannes Kepler described his theory of the mathematical movements of planets and wrote out the movement of planets in notes, arguing that the oval trajectory of the movement of planets creates melody (a musical illustration with “sounding” Saturn, Jupiter, Mars, Earth, Venus and Mercury was presented in his Book V, 207).

[15] The information was systematized by the author of this article – R. P.

Fibonacci Sequence: Historic-aesthetic Establishment

The considerations regarding numerical formulas and relationships was a point of permanent interest in the philosophical, aesthetic or academic sphere as well as on the practical plane, where the expression of proportions and / or composing according to them was regarded as the verification of perfect art work. For example, in the Renaissance the admiration for pure proportions of numbers were manifested as perspective in art, and striving for symmetry in architecture. Numerical proportions amply mentioned in the treatises on music during the Renaissance show that the mathematical aspect in music has survived as the main factor of beauty that is only slightly affected by the conception of sacrality.[16]

A significant role in creating music was attributed to the so-called infinite Fibonacci sequence – a mathematical proportion attributed to the mathematician of the 13th century Fibonacci (also referred to as Leonardo Pisano, Leonardo da Pisa, 1180–1250), who represented the principle of nature / evolution – the process of the reproduction of rabbits: how many young one pair of rabbits will produce per year, bearing in mind the fact that all the young will survive and reproduce further. Fibonacci, having summed up the number of rabbits of each month, obtained an infinite number progression 1–1–2–3–5–8–13–21 and so on.[17]

It should be noted, that there are many more earlier testimonies regarding the establishment of the famous number series. The most frequent reference is made to the scholar and philosopher of the Middle Ages, Boethius, as being the first to mention the Fibonacci sequence in the second book of his *De Institutione Arithmetica* (the beginning of the 5th c.). However, it was a translation of Nicomachus' *Introduction to Arithmetics*, written at the intersection of the 1st and 2nd centuries. In the second book of this treatise we may find an ancient reference to certain Fibonacci numbers. It is the description of the tenth neo-Pythagorean proportion, as Nicomachus argues:

> The tenth, in the full list, which concludes them all, and the fourth in the series presented by the moderns, is seen when, among three terms, as the mean is to the lesser,

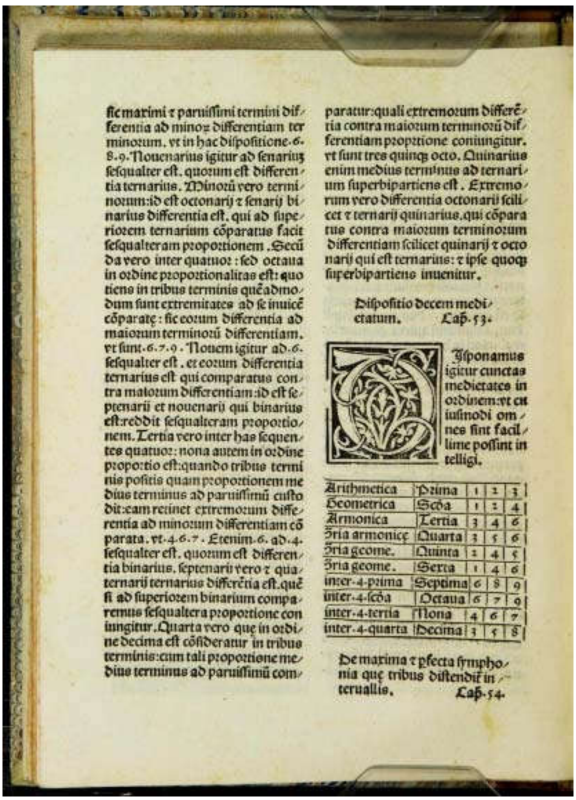
16 For example, in the treatise *Proportionale musices* (c. 1472–5) Johannes Tinctoris stated that proportions existed in everything, because "it was God who created them." He classified the proportions applied to the creation of music into five types: 1) the *genus multiplex* group made up of a formula n : 1 (2 : 1, 3 : 1, 4 : 1 and so on); 2) the *genus superparticularis* group, n + 1 : n (2 : 1, 3 : 2, 4 : 3 and so on); 3) the *genus superpartiens* group (n + 2 : n, n + 3 : n, n + 4 : n and so on); 4) the *genus multiplex superparticulare* group (n x m + 1 : n); 5) the *genus multiplex superpartiens* group (n x m + 2 : n, n x m + 3 : n, n x m + 4 : n and so on).

17 It is the recurrent sequence of the second order (denoted $F_1, F_2, \ldots, F_n$). The term $n + 2$ is equal to the sum of the two preceding terms (n and $n + 1$). The formula of progression: $F_{n+2} = F_{n+1} + F_n$, where $n > 0$.

so the difference of the extremes is to the difference of the greater terms, as 3, 5, 8, for it is the superbipartient ratio in each pair.[18]

Latin version by Boethius:

> *Quarta vero quae in ordine decima est consideratur in tribus terminis: cum tali proportione medius terminus ad parvissimum comparatur: quali extremorum differentia contra maiorum terminorum differentiam proportione coniungitur, ut sunt tres quinque octo. Quinarius enim medius terminus ad ternarium superbipartiens est. Extremorum vero differentia octonarij scilicet et ternarij quinarius qui comparatus contra maiorum terminorum differentiam scilicet quinarij et octonarij qui est ternarius: et ipse quoque superbipartiens invenitur.*[19]



sic maximi ⁊ paruissimi termini differentia ad minorum differentiam terminorum. vt in hac dispositione .6. 8.9. Nouenarius igitur ad senarium sesqualter est. quorum est differentia ternarius. Minorū vero terminorum: id est octonarij ⁊ senarij binarius differentia est. qui ad superiorem ternarium cōparatus facit sesqualteram proportionem. Secūda vero inter quatuor: sed octaua in ordine proportionalitas est: quotiens in tribus terminis quẽadmodum sunt extremitates ad se inuicẽ cōparatę: sic eorum differentia ad maiorum terminorū differentiam. vt sunt .6.7.9. Nouem igitur ad .6. sesqualter est. et eorum differentia ternarius est qui comparatus contra maiorum differentiam: id est septenarii et nouenarij qui binarius est: reddit sesqualteram proportionem. Tertia vero inter has sequentes quatuor: nona autem in ordine proportio est: quando tribus terminis positis quam proportionem medius terminus ad paruissimū custodit: eam retinet extremorum differentia ad minorum differentiam cōparata. vt .4.6.7. Etenim .6. ad .4. sesqualter est. quorum est differentia binarius. septenarij vero ⁊ quaternarij ternarius differẽtia est. quẽ si ad superiorem binarium comparemus sesqualtera proportione coniungitur. Quarta vero quę in ordine decima est cōsideratur in tribus terminis: cum tali proportione medius terminus ad paruissimū comparatur: quali extremorum differẽtia contra maiorum terminorū differentiam proportione coniungitur. vt sunt tres quinq; octo. Quinarius enim medius terminus ad ternarium superbipartiens est. Extremorum vero differentia octonarij scilicet ⁊ ternarij quinarius. qui cōparatus contra maiorum terminorum differentiam scilicet quinarij ⁊ octonarij qui est ternarius: ⁊ ipse quoq; superbipartiens inuenitur.

Dispositio decem medietatum. Cap. 53.

Disponamus igitur cunctas medietates in ordinem: vt eiusmodi omnes sint facillime possint intelligi.

Arithmetica	Prima	1	2	3
Geometrica	Scda	1	2	4
Armonica	Tertia	3	4	6
Ɔria armonicę	Quarta	3	5	6
Ɔria geome.	Quinta	2	4	5
Ɔria geome.	Sexta	1	4	6
inter .4. prima	Septima	6	8	9
inter .4. scda	Octaua	6	7	9
inter .4. tertia	Nona	4	6	7
inter .4. quarta	Decima	3	5	8

De maxima ⁊ pfecta symphonia quę tribus distendit interuallis. Cap. 54.

Example 2: Page extract from Boethius' *De Institutione Arithmetica* (1488) with the description of Nicomachus' tenth proportion

[18] Nicomachus of Gerasa, *Introduction to Arithmetic* (trans. Martin Luther D'Ooge), New York, The Macmillan Company, 1926, 284.

[19] *Arithmetica Boetij*, Augsburg: Erhard Ratdolt, 1488, Lap. 52.

Though, the origins of this number sequence date back to the 3rd century BC and may be attributed to Euclid's ideas about a certain relationship described in the sixth book of his *Elements* (c. 300 BC). Yet this special proportion, naming it an "extreme and mean ratio", was not analyzed as a particular row of numbers. The Greek mathematician relied on lines and inter-relationships between geometric constructions, as stated in the definition:

> A straight-line is said to have been cut in an extreme and mean ratio when as the whole is to the greater segment so is the greater (segment is) to the less.[20]

According to Ruth Tatlow, using a similar principle based on particular numbers in the 13th century, Fibonacci did not notice links between his discovery and Euclid's "special proportions" and did not denote that his numbers have "special relationships".[21] Moreover, the rules of the Fibonacci sequence may be applied to any sequence of numbers in which each new number is the sum of the two previous numbers. For example, the Lucas integer series, which was named after the 19th century French mathematician Édouard Lucas, and *Série Évangélique*, the latter based on the text of the Holy Bible:[22]

Lucas progression 2 – 1 – 3 (2 + 1) – 4 (1 + 3) – 7 (3 + 4) – 11–18 – 29–47 – 76–123 and so on

Série Évangélique 2 – 5 – 7 (2 + 5) – 12 (5 + 7) – 19 (7 + 12) – 31–50 – 81–131 and so on[23]

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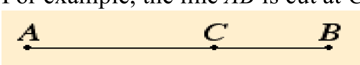
[20] *Euclid's Elements of Geometry*, edited and provided with a modern English translation by Richard Fitzpatrick, 2007, 156, http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf [2017-09-07].

For example, the line *AB* is cut at *C* since *AB* : *AC* = *AC* : *CB*:



[21] Ruth Tatlow, "The Use and Abuse of Fibonacci Numbers and the Golden Section in Musicology Today", in: *Understanding Bach*, Vol. 1, 2006, 77.

[22] The first numbers of *Série Évangélique*, 2, 5 and 12, were taken from the Gospel according to John (6, 9–13):

> There is a boy here who has five barley loaves and two fish, […] Jesus said, "Have the people sit down." Now there was much grass in the place. So, the men sat down, about five thousand in number. […] And when they had eaten their fill, he told his disciples, "Gather up the leftover fragments, that nothing may be lost." So they gathered them up and filled twelve baskets with fragments from the five barley loaves left by those who had eaten.

[23] It is important to note, that both progressions, Lucas and *Série Évangélique*, are directly derived from the numbers of Fibonacci sequence using the formulas F_{n+1} and F_{n-1} (For more information see: Newman W. Powel, "Fibonacci and the Gold Mean: Rumbas, Rabbits and Rondeaux", *Journal of Music Theory*, No. 23, 1979, 231).

Though what made the numerical series exclusive, which attracted the attention of thinkers and creators of different epochs? Perhaps the most important feature of the Fibonacci progression is the fact that by means of the ratio between them it is possible to come maximally close to the Golden Ratio formula, which expresses the laws of nature.[24] We may find a statement on it in the treatise *De Divina Proportione* (1509) by Luca Pacioli, naming this formula "the divine proportion". However, in 1890 Karl Fink indicated, that the establishment of Golden Section symbolism is attributed to astronomer Johannes Kepler, who "introduced the designation *sectio divina* as well as *proportio divina*" in the beginning of the 18th century:

> [Reciting Kepler] Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into extreme and mean ratio. The first we may compare to a mass of gold, the second we may call a precious jewel.[25]

As Fink states:

> The expression "golden section" is of more modern origin. It occurs in none of the text-books of the eighteenth century and appears to have been formed by a transfer from ordinary arithmetic. In the arithmetic of the sixteenth and seventeenth centuries the rule of three is frequently called the "golden rule". Since the beginning of the nineteenth century this golden rule has given way more and more before the so-called Schlussrechnen (analysis) of the Pestalozzi school. Consequently in place of the "golden rule", which is no longer known to the arithmetics, there appeared in the elementary geometries about the middle of the nineteenth century the "golden section", probably in connection with contemporary endeavors to attribute to this geometric construction the importance of a natural law.[26]

The conception of the Golden Ratio appeared in 1835 when the German mathematician Martin Ohm named the so-called "constant proportion" (German *stetige Proportion*) the *Goldener Schnitt*. Respectively, the very sequence of numbers was named after Fibonacci and was related to the phenomenon of the Golden Ratio only in the 19th century.[27]

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[24] Golden Ratio – Latin *sectio aurea*, *sectio divina*, German *Goldener Schnitt* – is expressed in terms of the formula *n x 0.618*. The fraction 0.618 was obtained having evaluated the ratios between the Fibonacci numbers. It became more accurate the more distant the pairs of Fibonacci numbers calculated, because 3 : 5 = 0.6; 5 : 8 = 0.625; 8 : 13 = 0.61538461…; 13 : 21 = 0.61904761…; 21 : 34 = 0.61764705… and so on.

[25] Karl Fink, *A Brief History of Mathematics*, (Transl. Wooster Woodruff Beman & David Eugene Smith), Chicago: The Open Court Publishing Company, 1990, 223. Original publication in German: Karl Fink, *Geschichte der Elementar-Mathematik*, 1890.

[26] Ibid., 223.

[27] In 1857, the Prince of Italy, the mathematician and historian Baldassarre Boncompagni, published a medieval treatise; in 1878, the French mathematician Edouard Lucas, having become acquainted with the treatise, named the said sequence the Fibonacci Sequence.

Fibonacci Manifestation in Music Composing

The Golden Ratio became the benchmark and the goal, and the guarantee, of perfect art. According to the measurements of the Golden Ratio, buildings were constructed, parks were planned, compositional details were applied to painting, and poetic stanzas were constructed. In music the formula of godly beauty symbolized a perfectly formed composition. Often the climax, the most important part of any composition, matched the point of the Golden Ratio. Sounds were organized and regulated according to Fibonacci principles of numeric progression, and so on. For example, while analyzing the structure of two vocal ballads from the Middle Ages, *Dame, se vous m'estes lonteinne* and *Je ne croy pas c'onques a creature,* by Guillaume de Machaut, Pozzi Escot shows that the limits of the structural parts adhere to the Golden Ratio (Escot 1999: 43, 51).

During the 19th century, which we may call the culmination of the antagonism between music and mathematics, where anti-rationalism was especially strong and where it would seem that emotionality was especially important, Fibonacci numbers and the Golden Ratio prevail nonetheless. Charles Madden indicates their presence in the structure of the first part of Beethoven's Fifth symphony,[28] as well as its characteristic for the structures of the compositions of Debussy.[29] The work by this French composer can be described as a medley of impressionistic tonal images and emotion. However, he has been quoted as stating:

> Music is the arithmetic of sounds as optics is the geometry of light.[30]

Roy Howat, while analyzing Debussy's *Dialogue du vent et de la mer* or the first piece *Reflets dans l'eau*, notes that the same wave in their structure can be seen in Katsushika Hokusai's colored woodcut *The Great Wave off Kanagawa* (1831), which illustrates the phenomenon of the Golden Ratio.[31] According to Allan W. Atlas, structural and harmonic changes are important in the duets, arias, and orchestral interludes of Puccini's opera *La boheme* and reflect the relationship of the Golden Ratio.[32]

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[28] Charles Madden, *Fib and Phi in Music: The Golden Proportion in Musical Form*, High Art Press, 2005.

[29] For example see: Roy Howat, *Debussy in Proportion*, Cambridge, 1983; Irina Soussidko, "Metrotektonik und Goldener Schnitt. Debussy 24 Preludes für Klavier", *Musik & Ästhetik*, 6. Jahrgang, Heft 24, 2002, 5–19.

[30] Dean Keith Simonton, *Greatness: Who Makes History and Why*, Guilford Press, 1994, 110.

[31] Roy Howat, op. cit., 23–9, 93–109.

[32] Allan W. Atlas, "Stealing a Kiss at the Golden Section: Pacing and Proportion in the Act I Love Duet of La Boheme", in: *Acta Musicologica*, Vol. 2, 2003, 269–291.

In the practice of contemporary music composition an especially active and conscious organization of sound space can be witnessed in the Fibonacci numbers. Most likely the attention of composers is drawn to this phenomenon by the opportunity to create asymmetry and irregularity in their music and to break mechanical proportions, because, as Valeria Cenova states, such a sequence allows music to "breathe".[33] The research has shown that the models of proportions are often used to organize the musical rhythm. Already in the beginning of the 20th century, composers accepted the Fibonacci sequence and the phenomenon of the Golden Ratio as a composition's standard of perfection. To paraphrase Emil Rozenov, even stylistically different musical compositions share the same quality – the manifestation of the Golden Ratio, which controls the music material as an expression of natural beauty.[34]

Ernő Lendvai also believed that the Golden Ratio is probably the most important aspect of a musical composition's architectonics and that it influences all of a composition's parameters. He researched and identified the impact of the Golden Ratio in the choice of climax, tone pitches, musical rhythm, grouping of measures, time signature, as well as proportions between a musical composition's parts, focusing on the work of Bartók, especially on his *Music for Strings, Percussion and Celesta* (1936).

As Diane Luchese states, another Hungarian composer Ligeti in his piece for organ *Volumina* (1961–2), applied the logic of the Golden Ratio to the duration of sections with different types of clusters (more see Luchese 1988). According to Ligeti himself, certain Fibonacci numbers are important to the structure of the first section of his *Apparitions* for orchestra (1958–9): in m. 71, at the 144th quarter note, as the bass plays *tremolo*, the first part is divided into two sections; in the second section a striking change in timbre occurs analogously in this sections' 55th quarter note – here an uninterrupted cluster comes in, which plays until the very end of the first part. The initial Fibonacci numbers, 1, 2, 3, 5, 8, and 13, also dominate the structure of Spanish composer Cristobal Halffter's *Fibonaciana*, Concert for flute and orchestra (1969).

How Fibonacci numbers can influence a composition's rhythmic design is illustrated by Sofia Gubaidulina's ensemble for percussion *In the Beginning there was Rhythm* (1984): the numbers 1, 2, 3, 5, 8 pop out in the rhythm of the kettledrum solo. In Gubaidulina's twelve-part composition for symphony orchestra *I hear… Silence…* (1986), in the odd-numbered parts I, III, V and VII, Fibonacci numbers dictate the time signatures accordingly: 144/4, 89/4, 55/4

[33] Валерия С. Ценова, *Числовые тайны музыки Софии Губайдулиной* [Numerical Secrets in the Music of Sofia Gubaidulina], Moscow, Moscow Conservatoire, 2000, 51.

[34] Эмилий Карлович Розенов, *Статьи о музыке* [Articles on Music], Moscow, 1982, 120–1.

and 34/4; in the part IX, as the composition's silent culmination – in the mute space the conductor's hands follow the rhythm according to the Fibonacci sequence (each of the gestures is schemed in detail by the composer).[35]

Lithuanian composer Osvaldas Balakauskas is regarded as "an individual who has been at the pinnacle of modern Lithuanian music since the mid-1960s, as well as being one of the most remarkable composers and leading authorities" and "belongs to the rare circle of composers whose ambition is the creation of their own precise musical system".[36] His Second symphony (1979) is an example of logic and rational musical thinking. For the segmentation of tone scale and the organization of the progressive rhythm the composer appllied a slightly transformed Fibonacci sequence with the first numbers 1, 2, 3, 5, 8, and 14 (the latter number does not belong in the sequence).[37]

Cases of Fibonacci numbers application

Prelude No. 1 in C major, Op. 28 (1838–9) by Frederic Chopin

The Fibonacci sequence may become the key to interpreting the structure of Frederic Chopin's (1810–1849) preludes as well.[38] In this article I would like to present an analysis of Prelude No. 1 in C major, Op. 28, that shows several instances in which Fibonacci numbers 8, 13, 21, and 34 make an appearance. This miniature composed for piano according to the form of the period is made of two phrases consisting of 8 and 16 measures, in addition to the establishment of C major in a 10 measure coda.

The analysis was focused on the significant musical signs (such as the highest/lowest pitch, climax, changes in harmony, etc.) taking into account the place of their appearance – particular measure. The following structural points revealed the existence of Fibonacci numbers:

1) in the 8th measure the lowest tone of the prelude is reached, the tone G_1 (contra-octave *sol*),

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[35] For more information see: Валерия С. Ценова, op. cit.

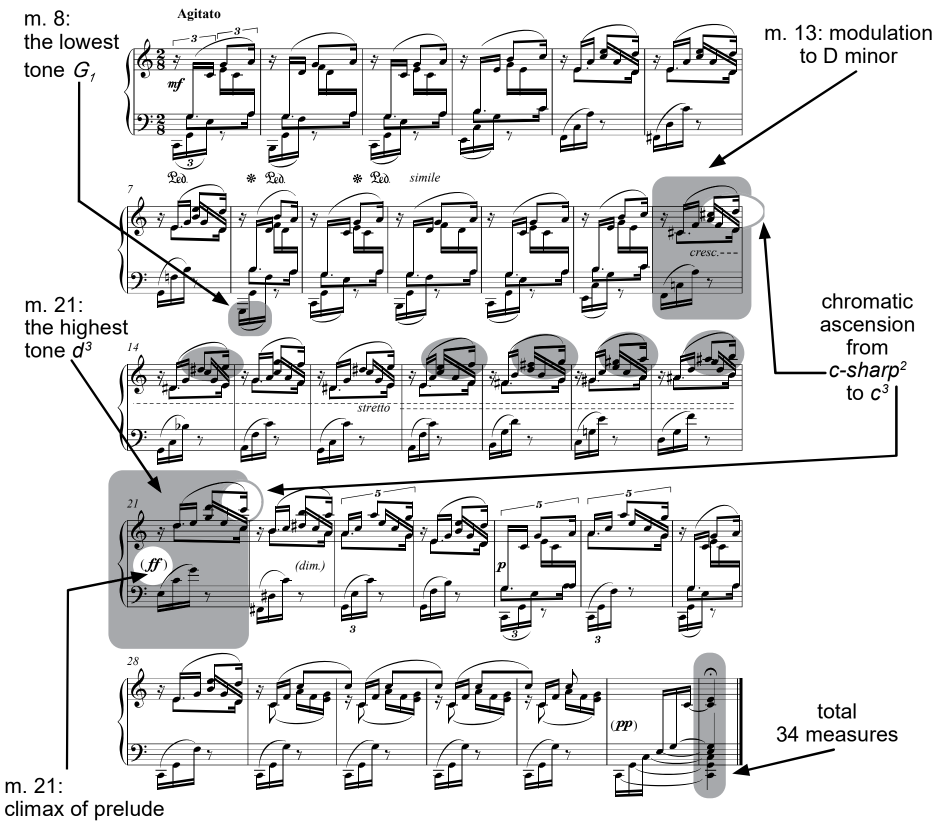
[36] More about Osvaldas Balakauskas see official Lithuanian composers' presentation by Music Information Centre Lithuania: http://www.mic.lt/en/database/classical/composers/balakauskas/.

[37] A comprehensive study on Balakauskas' work (*Osvaldas Balakauskas: Muzika ir mintys* [Music and Ideas], Rūta Gaidamavičiūtė (Ed.), Vilnius, Baltos lankos, 2000) presents a wide spectrum of his composing technique. On the Second symphony see especially an article by Gražina Daunoravičienė "Kompozicinės technikos algoritmai: Antroji simfonija ir Dada concerto" [Algorithms of Compositional Technique: The Second Symphony and Dada Concerto], 71–122.

[38] More about preludes see: Kenneth Patrick Kirk, *The Golden Ratio in Chopin's Preludes, Opus 28*, Doctoral Dissertation, Ann Arbor Mich, UMI, 2001.

2) in the 13th measure the first meaningful step towards harmony is taken, a modulation to D minor,
3) from the same measure (13) the diatonic picture of the composition is changed by the chromatic melody rising from *c-sharp*^{2} to *c*^{3} and then deepened by relief of a modulation,
4) in the 21st measure the highest tone of the prelude sounds, *d*^{3},
5) the 21st measure is the climax of the entire prelude,
6) in the mentioned measure the chromatic movement of the melody is changed by a diatonic movement downwards;
7) the prelude is made up of 34 measures.

Prelude analysis is presented in the Example 3, highlighting the points of Fibonacci numbers performance.



Example 3: Structural analysis of Chopin's Prelude No. 1 C major, Op. 28

Derek Bourgeois. Symphony for Organ, Op. 48 (1975)

The especially prolific English composer Derek Bourgeois (born 1941) dedicated the third part of his Symphony for Organ (1975) to Fibonacci and titled it *Passacaglia di Fibonacci*. Having completed an analysis of the score, it became apparent that the various parameters of the composition were influenced by the Fibonacci numerical algorithm. They are the following:

1) the entire part is made up of 144 measures,

2) the composer uses 13 different time signatures,

3) the time signatures are written using Fibonacci numbers:

1/8, 2/8, 3/8, 5/8, 8/8, 13/8, etc.

4) the third part has inscriptions of five tempo markings,

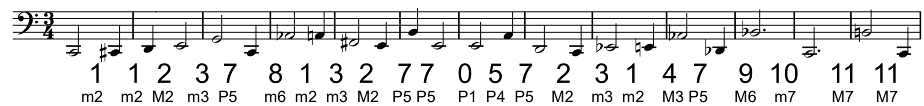
5) tempo marking *Largo maestoso* lasts 8 measures,

6) marking *l'isteso tempo* lasts 5 measures,

7) the 24-tone subject's length in measures is 13.

The subject is continually repeated as *basso ostinato*. The arrangement of its melody-line in semitones is worth its own discussion, because an internal logic at first is hardly noticeable and sounds even chaotic. Firstly it is suggested to transcribe the intervals of melody-line into numerical row. Therefore we get a number sequence as follows:

1–1–2–3–7–8–1–3–2–7–7–0–5–7–2–3–1–4–7–9–10–11–11



Example 4: 24-tone subject and its numerical transcription. Bourgeois, Symphony for Organ, Op. 48, part 3 *Passacaglia di Fibonacci*, mm. 14–28

At a first glance the row looks like an irrational interchange of certain numbers. But in fact the melody is constructed according to the "hidden" symmetrical relationship between tones, which is explained with the help of Fibonacci numbers. Some members in the row above (like 4, 7, 9, 10 and 11) are not of the Fibonacci sequence. That is because the range in semitones of a few intervals does not match the Fibonacci numbers – that is the major third (M3, its range in semitones is 4), the fifth (P5, 7 semitones), the major sixth (M6, 9 semitones), and the minor and major sevenths (m7, 10 semitones, & M7, 11 semitones). However, the inversion of these music intervals and respectively the new range in semitones matches the Fibonacci numbers. Because, the M3 inversion is a minor sixth, whose range is 8 semitones; correspondingly, the P5 inversion is a perfect fourth and the range in semitones is 5; M6 inversion is a minor third (3 semitones); m7 inversion is major second and 2 semitones; and finally M7 inversion is a minor second and 1 semitone. Moreover, the numerical sequence, rewritten in Fibonacci numbers, exposes a mirror symmetry, and its center is marked by the only perfect unison with zero:

1	1	2	3	5	8	1	3	2	5	5	**0**	5	5	2	3	1	8	5	3	2	1	1
				(7)					(7)	(7)			(7)				(4)	(7)	(9)	(10)	(11)	(11)
m2	m2	M2	m3	P4	m6	m2	m3	M2	P4	P4	P1	P4	P4	M2	m3	m2	m6	P4	m3	M2	m2	m2
				P5					P5	P5			P5				M3	P5	M6	m7	M7	M7
				inverted interval					inverted interval	inverted interval			inverted interval				inverted interval	inverted interval	inverted interval	inverted interval	inverted interval	inverted interval

Example 5: Arrangement of musical intervals of the 24-tone subject according to Fibonacci numbers

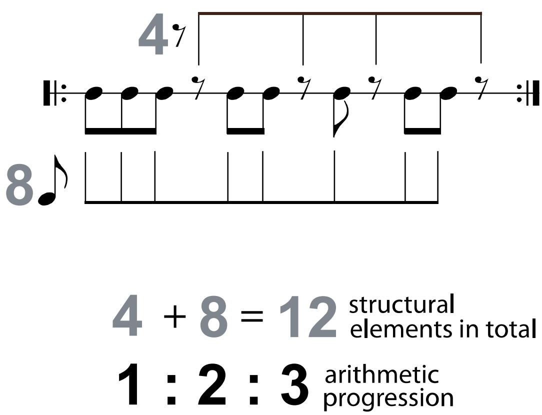
This observation allows one to surmise that the composer did not seek to use Fibonacci numbers in an obvious manner, but rather in a creative one. A similar incentive to creativity can be explained by the impact of the Fibonacci numbers on the time parameter: the number of measures are not recorded as usual (every 5 or 10 measures), but in those parts of the score that correlate with Fibonacci numbers. Bourgeois had an even cleverer idea by trying to hide Fibonacci numbers, by looking at the logic of sections 3, 4, and 5. Their volume in measures (respectively 65, 24, and 10) does not correspond to the Fibonacci numbers. However, if one were to remove the number 10 (extent of section 5) from the number of measures of section 3 (65 measures) or if one were to add the number 10 to the section 4 (24 measures), we would get Fibonacci numbers 55 and 34:

Sections	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	
Meter signatures	8/8	5/8	3/4	5/4	8/4	13/4	3/4	21/4	13/4	8/4	5/4	3/4	2/4	1/4	1/8	2/8	3/8	5/8	8/8	13/8	
Number of bars	**8**	**5**	*65*	*24*	*10*	**3**	**13**	**3**	**2**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**Total 144**
			– *10* = **55**	+ *10* = **34**	↑ (10 added to section 4)																

Example 6: Arrangement of time signatures and number of measures of sections according to Fibonacci numbers in Bourgeois' Symphony for organ, part 3 *Passacaglia di Fibonacci*

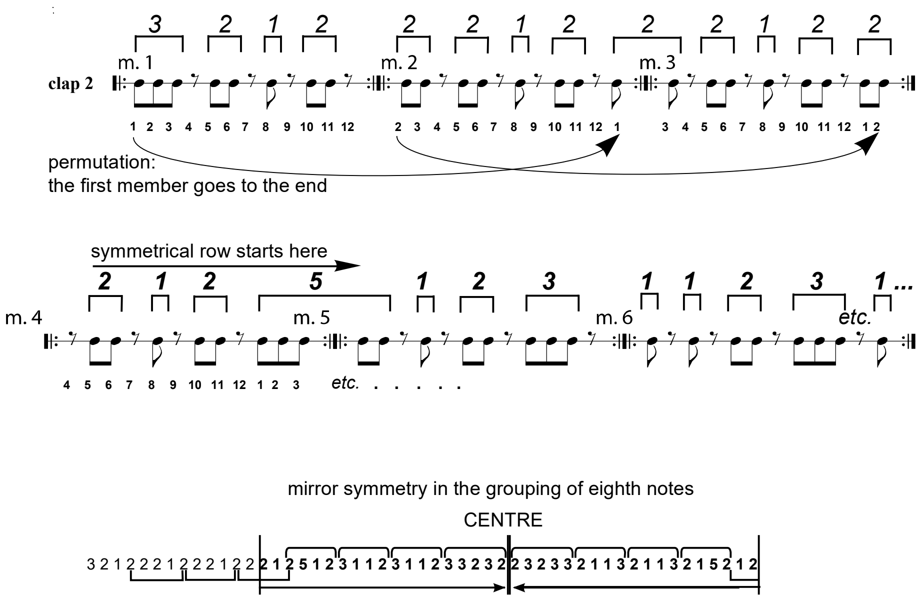
Steve Reich. *Clapping Music (1972)*

I would like to argue that one of the approaches analyzing the structure of Reich's *Clapping Music* (1972) could be the application of Fibonacci numbers, arithmetic progression and the laws of symmetry. This compact piece, first of all, sounds like a diversity of complex rhythmic models, little by little moving from the overlapping accents to the chaotic disagreement of two clapping performers. However, the rhythmic complexity is directed by a strict inner logic. For example, the arithmetic progression is characteristic for the structure of the basic rhythmic pattern, which the first performer claps consistently. It is because the tones, rests, and total number of elements are in the relationship of 4 : 8 : 12, where 4 is the sum of eighth note rests, 8 is the sum of eighth notes, and 12 is the total sum of musical elements; accordingly → 1 : 2 : 3.



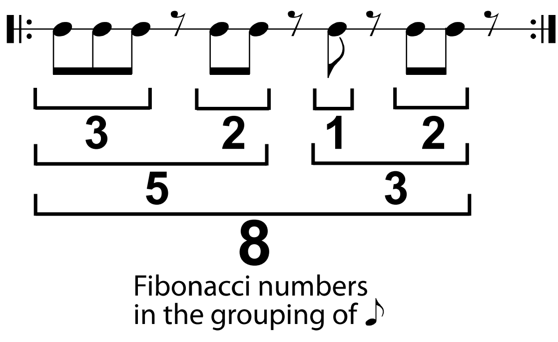
Example 7: Arrangement of basic rhythmic pattern according to arithmetic progression, Reich's *Clapping Music*

If we have a look at the second performer's score, one can see the manifestation of mirror symmetry. This part of the piece is composed of a basic rhythmic pattern and 11 of its cyclic permutations. The permutations are created consistently by carrying the first member to the end. The symmetry was established in the number sequence, which was written down after having counted the number of clapping tones of the second performer from one rest to the next. The symmetrical number row is presented in a graphic diagram (see Example 8).



Example 8: Manifestation of mirror symmetry in the part of the 2nd performer (the numbers indicate the quantity of clapping tones between the rests), Reich's *Clapping Music*

What about the possible manifestation of Fibonacci logic? Let us look at the rhythmic invariant of the piece that is made up of 8 eighth notes. The eighth rests divide the row of notes into four sections according to the following: 3 + 2 + 1 + 2. Thus we get a combination of the first Fibonacci numbers 1, 2, 3, (5), and 8 (see Example 9).



Example 9: Exposition of Fibonacci numbers, analyzing the basic rhythmic pattern in Reich's *Clapping Music*

Could Fibonacci numbers influence other structural parameters as well? If we have a look at the length of the piece, we see a total of 13 measures, of which 12 are different from each other, plus at the end, the composer indicates the need to repeat bar 1. Further. I would make an observation regarding time signatures. A widely used 1980 Universal Edition printed score provides tempo inscriptions – numbers 160 and 184. However, the numbers are different compared to the very first intention. Because the 1972 December (re-copied in January 1978) manuscript included two time signatures 144 and 168. Evidently the first one is the 12th member of the Fibonacci row (F_{12}), whereas the second time signature – 168 – does not belong to the Fibonacci numbers, but... is a result of the multiplication of two members F_6 and F_8, because 168 = 8 x 21. Moreover, according to the directions for performance, the duration of the piece is around 5 minutes (once again the Fibonacci number).

Finally, an incentive for a sophisticated joke manipulating with Fibonacci numbers appears in the title *Clapping Music*. It is because the sum of letters is 13, and each word is made up of 8 or 5 letters. Here, I would end the search for Fibonacci and other mathematical phenomena in music with the reference to the Renaissance concept of *homo sapiens* as *homo ludens* (Latin – the playing man). Johan Huizinga who published the book *Homo Ludens* (*The Playing Man*) in 1938 thoroughly investigated the concept of "the playing man" as the concept of the theory of the game. Hermann Hesse wrote the following about the rule of the game:

> [T]he Glass Bead Game player plays like the organist on an organ. And this organ has attained an almost unimaginable perfection; its manuals and pedals range over the entire intellectual cosmos; its stops are almost beyond number. Theoretically this instrument is capable of reproducing in the Game the entire intellectual content of the universe.[39]

The application of various numerical operations and calculations to a music composition – why not an interesting and sophisticated game, that proves the intersection of music and mathematics?

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[39] Hermann Hesse, *Magister Ludi. The Glass Bead Game* (transl. Richard and Clara Winston), New York, Bantam Books, 1969, 6.

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The article based on material published in: Rima Povilionienė, *Musica Mathematica: Traditions and Innovations in Contemporary Music*. Series Methodology of Music Research, Vol. 9, ed. Nico Schuler. Frankfurt am Main, New York: Peter Lang Academic Research, 2016, 288 p. ISBN 978-3-631-71381-5, ISSN 1618-842X.

Summary

In this paper the author considers an interdisciplinary dialogue between music and mathematics, emphasizing the role of the Fibonacci principles in the context of musical compositions. The article presents three analytical cases – Chopin's piano prelude, Bourgeois' composition for organ and Reich's minimalistic piece, attempting to render music composition structure to the logic of Fibonacci numbers.